The Tsallis Parameter

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Abstract

The exact solution of a particular form of the stationary state generalized Fokker-Planck equations, which is given under certain conditions by the classical Tsallis distribution, is compared with the solution of the MAXENT equations obtained using the classical Tsallis entropy. The solutions only agree provided the Tsallis parameter, q, is no longer taken to be constant.

The nonextensive entropic measure proposed by Tsallis[1, 2] introduces a parameter, q, which is not defined[3, 4]. It has been argued that perhaps this parameter should be constant, if not universally, at least for classes of dynamical systems[5–7]. Attempts have been made to set limits on the value of this parameter [8]. For the practitioners this has been accepted de facto and calculations involving the Tsallis entropy have generally used one piece of data to determine the value of q which is then fixed over the range of interest[5, 8]. However, for the generalized Fokker-Planck(GFP) equation[9], dissipative optical lattices[10] and more recently in the analysis of relativistic heavy ion collision [11] q is most definitely not a constant, unless as in the latter case, complicated scaling arguments are invoked.

In the present letter we consider the GFP equations [12] where an exact solution (both static and time dependent) has been found and is shown under certain circumstances to be equivalent to the Tsallis classical distribution functions. In the static case we formulate and solve the equivalent MAXENT problem[13] for different values of the equation of constraint. We solve for q in a manner suggested earlier[14] and find the numerical dependence of q on the Lagrange multiplier associated with the equation of constraint.

Consider the following non-linear one dimensional GFP equation[9]

$$\frac{\partial F}{\partial t} = -\frac{\partial}{\partial x}\{K(x)F\} + \frac{1}{2}Q\frac{\partial^2[F^{2-q}]}{\partial x^2}, \tag{1}$$

where F is the distribution function, Q is the diffusion coefficient, and K(x) is the drift coefficient which determines the potential:

$$V(x) = -\int_{x_0}^{x} K(x)dx. \tag{2}$$

The particular power $q-2$ is chosen in accordance with the quite general discussion of the generalized Bogulubov inequalities[15] which points out that systems which obey Tsallis statistics exhibit abrupt changes at $q = 2$.

Assuming the boundary conditions $F \to 0$ for $x \to \pm\infty$, the differential equation governing stationary solutions ($\frac{\partial F}{\partial t} = 0$) can be written as

$$K(x)F = \frac{1}{2}Q\frac{\partial(F^{2-q})}{\partial x}. \tag{3}$$

By performing the redefinition $u = F^{2-q}$, separating variables and integrating, one obtains the following stationary solution

$$F(x) = D[1 - \beta(1-q)V(x)]^{\frac{1}{1-q}}, \tag{4}$$

where D is a positive (undetermined) integration constant. Eq. (4) is precisely the form of the distribution function obtained from Tsallis Statistics[1, 9]. One can verify by substitution that Eq.(4) is a solution to Eq. (3) provided

$$\beta = \frac{2}{Q}[\frac{D^{q-1}}{2-q}] \tag{5}$$

Clearly, β is a function of q. It should be noted that this distribution need not be normalized to 1. Also, because the GFP Equation is nonlinear, for a particular value of β and q there is only one value of D that solves Eq. (3). Therefore, D is also an (undetermined) function of β and q. Note that for $F(x) > 0$, $\beta > 0$ and q$<$ 2. In the limit $q \to 1$ the GFP reduces to the standard Fokker-Planck equation, $\beta \to \frac{2}{Q}$ and the stationary state adopts the usual form

$$F(x) = Dexp[-(2/Q)V(x) \tag{6}$$

The stationary state distribution can also be obtained from the classical Tsallis entropy

$$S_q = \frac{1}{q-1}(F - F^q) \tag{7}$$

via the MAXENT equation[13]

$$\delta_F S_q = 0 \tag{8}$$

along with the equation of constraint

$$Tr[F^q V] = C \tag{9}$$

The solution of these equations yields the classical Tsallis distribution given in Eq.(4) for any allowed value of q. In order to simultaneously determine q an additional equation

$$\frac{\partial S}{\partial q}|_\beta = 0 \tag{10}$$

must be solved[14]. Note the distribution obtained is not normalized.

For the simple case where V is a constant, the phase space integrals in the trace each produce a constant factor which can be absorbed into C, yielding

$$F^q V = C \tag{11}$$

By solving Eq. (11) for β and substituting it into Eq. (8), we numerically solve Eq. (10) for q. FIG. (1) is a plot of β vs. q for V = 1.

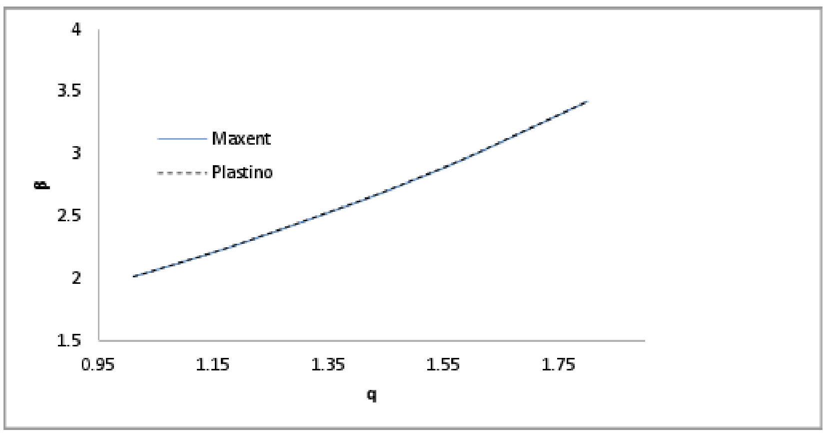

FIG. 1: β vs. q for V=1, and $\beta = \frac{2}{Q} D_o^{q-1}$

This curve can be fitted by the function $\beta = \frac{2}{Q} D_o^{q-1}$, where $D_o = 1.96$ and Q=1. As already mentioned, D in Eq. (5) is an undetermined function of β, and therefore q. To match the MaxEnt and GFP solutions, one must identify $D = D_o(2-q)^{\frac{1}{q-1}}$.

Since q is a variable one cannot choose its value from one data point and assume it remains fixed over the entire range of interest. To further illustrate this point consider $C(\beta)$ as given by Eq. (11), for constant V. For a thermodynamic system this can be thought of

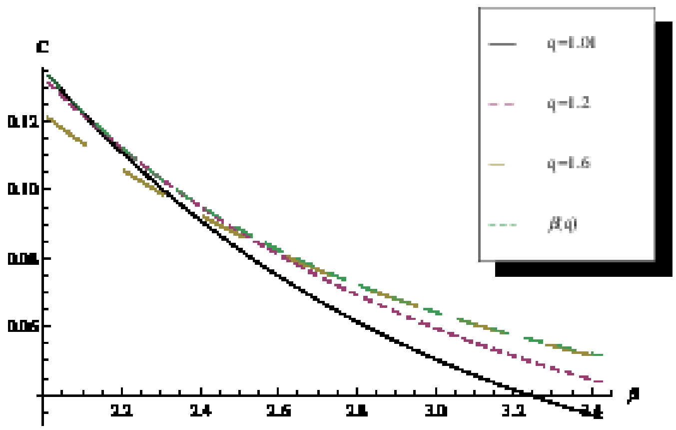

FIG. 2: C vs. β for V=1, $\beta(q) = \frac{2}{Q} D_o^{q-1}$

as the thermal response of the system. FIG. (2) plots C vs. β for fixed values of q as well as for $\beta = \frac{2}{Q} D_o^{q-1}$. Clearly the results obtained both in magnitude and shape do not agree well except in the region around the value of β corresponding fixed chosen value of q.

Our results indicate that some care should be taken when the Maxent equations are used with the Tsallis entropic measure to describe experimental data. Although we have only considered the stationary solutions of a particular form of the one dimensional GFP equation, the MAXENT solution is only consistent with the exact solution when q is no longer treated as a constant. Similar behavior is also to be expected in the solutions of the time dependent GFP equation. Even if a scaling function can be found for every dynamical situation, the functional dependence of the unscaled Tsallis parameter should be determined as a function of any Lagrange mutipliers associated with the equations of constraint.

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